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Review

Arts revealed in calculus and its extension

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Motivated by presenting mathematics visually and interestingly to common people based on calculus and its extension, parametric curves are explored here to have two and three dimensional objects such that these objects can be used for demonstrating mathematics.
Epicycloid, hypocycloid are particular curves that are implemented in MATLAB programs and the motifs are presented here. The obtained curves are considered to be domains for complex mappings to have new variation of Figures and objects. Additionally Voronoi mapping is also implemented to some parametric curves and some resulting complex mappings.
Some obtained 3 dimensional objects are considered as flowers and animals inspiring to be mathematical ornaments of hypocycloid dance which is also illustrated here.

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INTRODUCTION

Real functions have been introduced in schools such as polynomials, trigonometry, rational functions and mixtures of those functions on real domains. On the other hand, there are only few topics on complex domain related to a complex number, its algebra and a complex function. These are not introduced in senior high schools since its complexity for delivering its idea. This leads to a big gap knowledge in mathematics for students if students pose problems on a complex function and its analysis that may arise in many application of mathematics. Teachers may give an overview of a complex number, its algebra and functions on a complex domain. This paper addresses on visualization of complex mapping on a complex domain defined by parametric equations.
There are some well known parametric curves known in calculus, e.g. cycloid, hypocycloid and epitrochoid. These curves are recalled here to have new invention of arts in calculus by varying the values of parameters. The mathematical representation of cycloid, hypocycloid and epitrochoid respectively are

$$x(t) = a\sin(t - \sin t);\ y(t) = a(t - \cos t) \tag{1.1}$$

$$x(t) = (a-b)\cos t + b\cos\left(\left(\frac{a-b}{b}\right)t\right);\quad y(t) = (a-b)\sin t + b\sin\left(\left(\frac{a-b}{b}\right)t\right); \tag{1.2}$$

$$x(t) = (a-b)\cos t + c\cos\left(\left(\frac{a}{b}-1\right)t\right);\quad y(t) = (a-b)\sin t + c\sin\left(\left(\frac{a}{b}-1\right)t\right). \tag{1.3}$$

The hypocycloid curve produced by a fixed point P on the circumference of a small circle of radius *b* rolling around the inside of a large circle of radius *a* (*a* >*b*), where *t* the angular displacement of the center of small circle (Hsu,et.all,2008). Many authors considered the resulting curves due to integers value of *a*/*b*. Designing cycloid reducers and designing an example for a rotating ring gear type epicycloid reducer (Shin) and designing hypocycloid gear assembly used in internal combustion engine are particular examples. By considering the ratio of *a/b* can be any irrational values, one may observe different types of curves as shown in Figure 1. These special curves are drawn with different styles of line ,colors and different values of parameter to obtain some variant of these curves. Now, one may use different color for the used lines and types of lines to draw the basic pattern as shown in these examples.

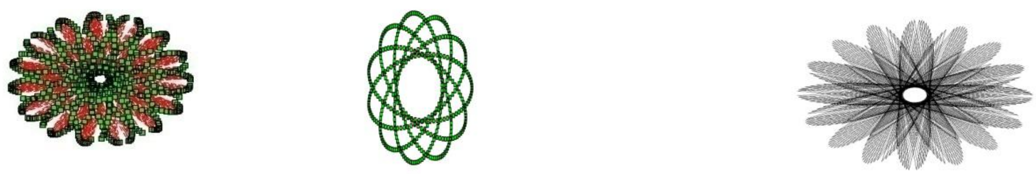

Figure 1. From left to right : hypocycloid with $a=\pi, b=\sqrt{2}, 0\le t\le 200\pi;$ epitrochoid with $a=\pi, b=\sqrt{2}, c=5, 0\le t\le 15\pi$; hypocycloid with $a=0, b=\pi, 0\le t\le 250\pi.$

The research designs different motifs obtained from the parametric curves above and using different types of mapping such as complex mapping and Voronoi mapping. Some variants in colours, lines and other possibilities are also used. Kinds of flower and animals appear in these motifs inspiring to design three dimensional objects used as properties of hypocycloid dance which is already available in the youtube (its link keyword : hypocycloid dance).
This paper shows some standard ideas for presenting parametric curves mapped by complex mapping and Voronoi mapping and the used basic theories are presented in the second chapter.

COMPLEX MAPPING AND VORONOI MAPPING

Complex Mapping

Complex mappings such as conformal and noncormal mappings have good features for new creation of the motifs above. The basic idea is that all points creating motifs above are considered as complex domains and mapped by the chosen conformal or nonconformal mappings. Some basic visualization of these maps have been studied (Parhusip,2010) for instance: 1/*z* and $(1/z)^{\alpha}$ with $\alpha\in[-1,1]$ which have been proven as conformal mappings (Parhusip and Sulistyono,2009). Colors also play important roles to have good images for the new motifs. Note that each range of conformal/nonconformal graph is obtained by expressing real part and imaginary part of the mapping results to the program explicitly. Several complex functions have been explicitly defined the real parts and imaginary parts (Parhusip,2010). Some examples are shown here.

Example 1. Let us study the function *w*=sin(*z*). The usual way for drawing it is separating the real part and the imaginary part. If *w* = sin(*x* + *iy*) = sin*x* cos(i*y*) + cos *x* sin(i*y*) then we have no information how cosine and sine work for a complex number. Thank to Euler, we define $e^{i\theta}=\cos\theta+i\sin\theta$. We directly assume that $e^{iz}=\cos z+i\sin z$ and $e^{-iz}=\cos z-i\sin z$. By adding and subtracting both respectively we get

$$\frac{e^{iz}+e^{-iz}}{2}=\cos z \text{ and } \frac{e^{iz}-e^{-iz}}{2}=\sin z. \qquad (2)$$

Again, the real part and the imaginary part are not shown explicitly and hence we need the definition
$e^{iz}=e^{i(x+iy)}=e^{-y}e^{ix}$ and $e^{-iz}=e^{-i(x+iy)}=e^{y}e^{-ix}$.
Using these expressions to Equation (2) one yields

$$\sin z=\frac{1}{2}\left(e^{y}+e^{-y}\right)\sin x+\frac{i}{2}\left(e^{y}+e^{-y}\right)\cos x=\sin x\cosh y+i\cos x\sinh y$$

Thus the real part and imaginary part of sin (z) respectively are

$u(x,y)=\frac{1}{2}\left(e^{y}+e^{-y}\right)\sin x$ and $v(x,y)=\frac{1}{2}\left(e^{y}+e^{-y}\right)\cos x$.

Example 2.
The function cos z is defined by Equation (2). Proceeding as above, one yields the real part and the imaginary part of cos(z),i.e
$u(x,y)=\frac{1}{2}\left(e^{y}+e^{-y}\right)\cos x$ and $v(x,y)=\frac{1}{2}\left(e^{-y}-e^{y}\right)\sin x$.

Example 3.

$$w=\frac{1}{1-z}=\frac{1}{1-(x+iy)}=\frac{1}{(1-x)-iy}\times\frac{(1-x)+iy}{(1-x)+iy}=\frac{(1-x)+iy}{(1-x)^{2}+y^{2}}$$

Therefore
$u(x,y)=\frac{(1-x)}{(1-x)^{2}+y^{2}}$ and $v(x,y)=\frac{y}{(1-x)^{2}+y^{2}}$.

Example 4.
Consider $w=\frac{1+z}{1-z}$ We obtain $u(x,y)=\frac{1-x^{2}-y^{2}}{(1-x)^{2}+y^{2}}$ and $v(x,y)=\frac{2y}{(1-x)^{2}+y^{2}}$

Visualization on Derivative of a Complex Mapping

Derivative of a complex function relies on differentiability of this complex function in a complex domain, i.e *f* is holomorphic. Some basic rules of differentiability of complex functions are actually similar in real cases. If we have $f(z)=z^{n}$ with n a positive integer, one may prove that $\frac{df}{dz}=n\,z^{n-1}$.
Here, we will not discuss into detail about differentiability of a complex function. Our interest is that how derivative of a complex function can be visualized. One possibility is that after having all points of the resulting complex mapping, the derivatives of these all points are mapped again by the derivative of this complex function.

Example 5.
One may prove that if $w=\frac{1+z}{1-z}$ then $\frac{dw}{dz}=\frac{2}{(1-z)^{2}}$ .Hence $\frac{dw}{dz}$ is not analytic or singular on z=1.
We should have

$$\frac{1}{(1-z)^{2}}=\frac{1}{1-2z+z^{2}}=\frac{1}{1-2(x+iy)+(x+iy)^{2}}=\frac{1}{1-2x+x^{2}-y^{2}+(2xy-2y)i}$$

$$:=\frac{1}{p(x,y)+iq(x,y)}$$

By multiplying the equation with $\frac{p-iq}{p-iq}$, one yields $\frac{dw}{dz}=2\frac{p-qi}{p^{2}+q^{2}}$.

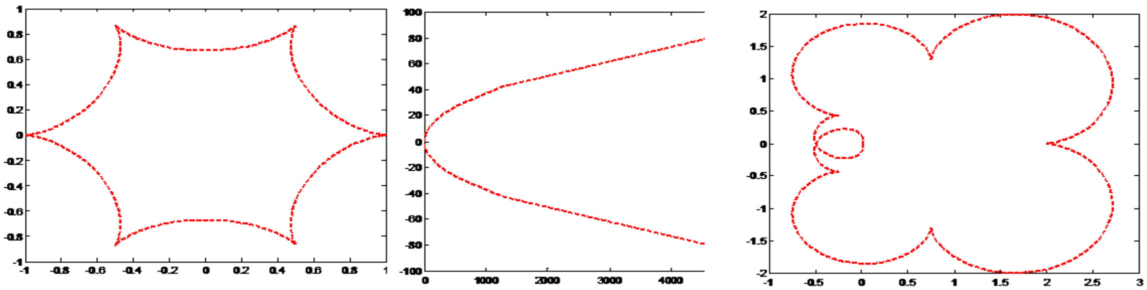

Figure 2. From left to right :6-cusp hypocycloid with *a*=1,*a*/*b*=5, $0 \le t \le 2/\pi$; mapping of *w* in example 4 (middle) and its derivative (right).

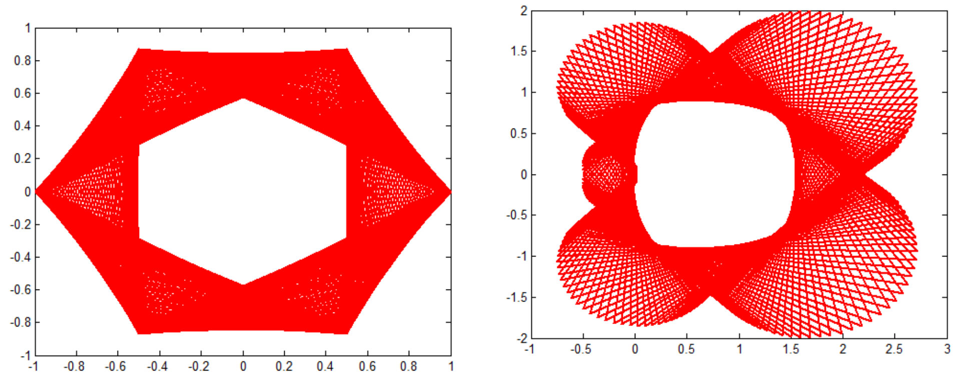

Figure 3. From left to right :6-cusp hypocycloid with *a*=1,*a*/*b*=5, $0 \le t \le 200\pi$; mapping of *w* in example 4 (middle) and its derivative (right).

Table 1. Program MATLAB to visualize hypocycloid and its complex mapping by 1/*z*.

Program of hypocycloid	(continuing left program)
%hypocycloid	plot(xh,yh,'--rs','LineWidth',2,...
clear	'MarkerEdgeColor','g',...
close all	'MarkerFaceColor','g',...
%hypocycloid	'MarkerSize',10)
ah=pi;bh=sqrt(2);	%f(z)=1/z transformation
a3=0;b3=200*pi;	below=xh.^2+yh.^2;
t3=linspace(a3,b3,500);	u=xh./below;
rasio=(ah-bh)/bh;	v=-yh./below;
xh=(ah-bh)*cos(t3)+ bh*cos(rasio*t3);	figure
yh=(ah-bh)*sin(t3)-bh*sin(rasio*t3);	plot(u,v,'--ro','LineWidth',2,...
figure	'MarkerEdgeColor','r',...
	'MarkerFaceColor','k',...
	'MarkerSize',10)

Readers are suggested to explore this idea for many other complex functions that are difficult to visualize in the ages of traditional teaching (see for instance: based on the book Schaum series). Furthermore, one may vary other parameters in Equation (1.2) to have an art sense of this derivative. The example of the visualization is illustrated in Figure 3. The mapping result is attractive for the purpose of finding a good design of motif. However its derivative is considered an attractive enough to pose here.

The parametric curves are programmed in MATLAB and the results are considered to be complex domains mapped by complex functions. The hypocycloid shown by Figure 1 is modified by including a complex mapping in the program. A small program is shown in Table 1 here to encourage readers in this field.

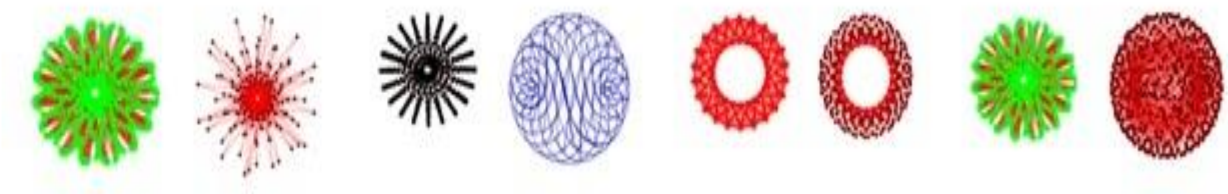

Figure 4. Each pair contains a parametric curve and its complex mapping

Voronoi Mapping

Voronoi mapping designs a set *P* of of distinct points $p_1, p_2,..., p_n$ in the Euclidean plane into regions $R(p_i)$, *i*=1,…,*n* associated with each member of *P*, such that each point in a region $R(p_i)$, *i*=1,…,*n* is closer to p_i than any other point in *P* (Patel,-). This mapping is frequently discussed in computational geometry known as Voronoi diagram and it is the geometric dual (Delaunay Triangulation). Available introduction on Voronoi diagram may be found through internet that one may have easily such as Fundamental of Voronoi diagram and its used in application had been explained (Aurenhammer,1991). A variant of map algebra using Voronoi diagram for instance, has been successfully created for two and three dimensions (Ledoux and Gold ,2006). This research is not building a Voronoi diagram, it is only using the generator of Voronoi generator provided by MATLAB. Instead of constructing points to be a set for a Voronoi diagram, all points are designed by parametric curves obtained above.

In this paper, one uses some domains obtained by parametric curves and some domains obtained by complex mapping of parametric curves which are mapped by Voronoi mappings. Therefore, one may have directly results of Voronoi mapping as soon as the domains are determined.

RESULT AND DISCUSSION

Results on Complex Mappings

There are several parametric curves can be mapped by complex mappings. Some complex functions are used here, e.g $1/z, z^2, \cos z, \sin z, e^z$. The mapping results are obtained and shown in Figure 4 where each pair illustrates a parametric curve and its mapping by a complex function. Composition of functions may also be defined to have a new creation. One of examples is Figure 4. The first left Figure4 illustrates the hypocycloid Equation (1.2) using Golden ratio, i.e the value of $\frac{a+b}{b} = 1.6$ and $b = \sqrt{2}$. It is expected that this ratio gives better creation in the obtained figure. The second figure is the mapping result of f(*z*)=1/*z* and the third is its composite with cos *z*. This figure is again mapped by cos *z*, and one yields the fourth in Figure 5.

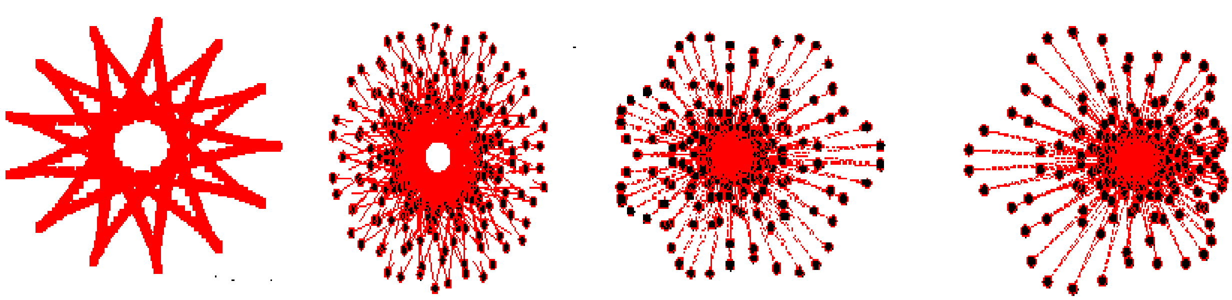

Figure 5. From left to right: Hypocycloid whose its ratio satisfies Golden ratio, with $b = \sqrt{2}$ then mapped by *f*(*z*)=1/*z* (2-nd), composite with cos *z*) (3-rd) and again with cos z (4-th).

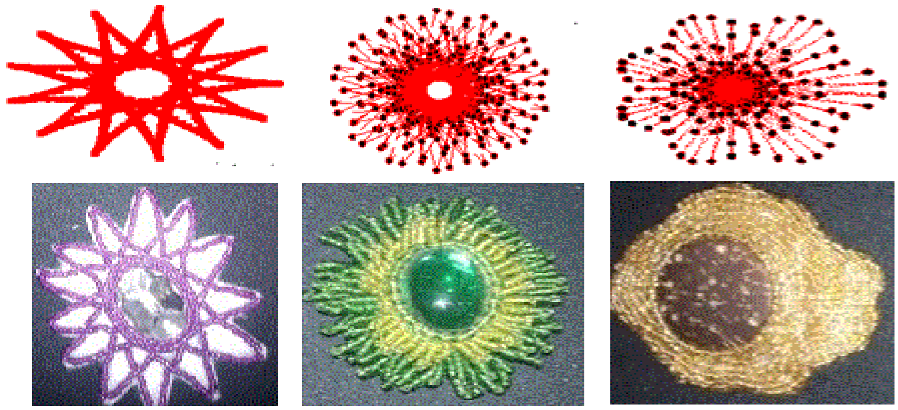

Figure 6. From left to right: Each pair (above and below) illustrates the motif and its ornament.

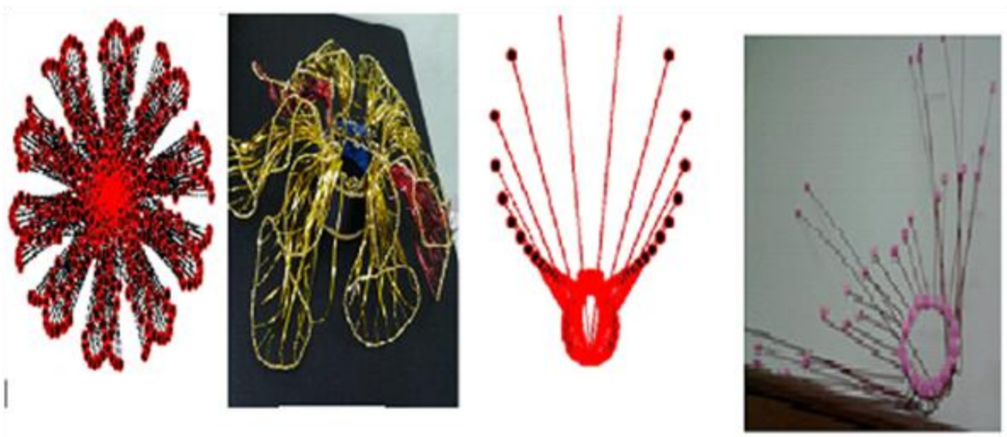

Figure 7. From left to right: Each pair illustrates the motif and its ornament.

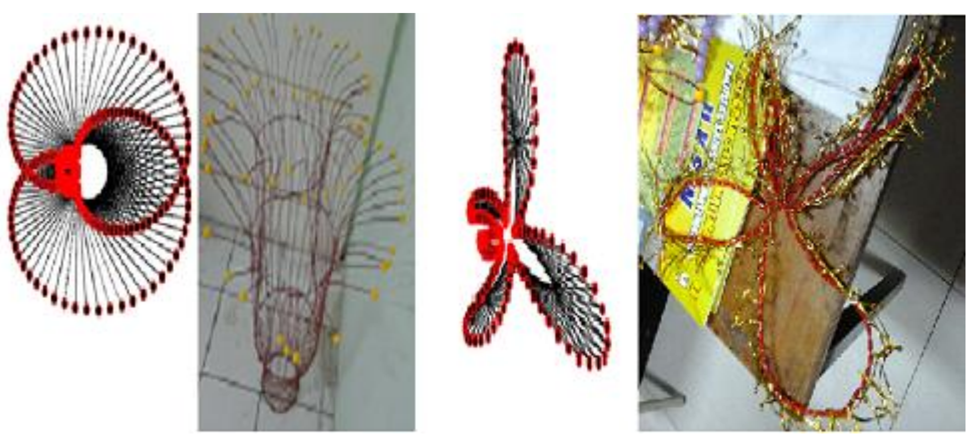

Figure 8. From left to right: First pair is a hypocycloid (Equation (1.2) mapped by $f(z)=\cos z$.
Second pair :$x(t)=r\cos(2\pi t)$;$y(t)=r\sin(2\pi t)$; $r=2+\sin(20\pi t)$; $0 \le t \le \pi/2$.

The motifs are used to design some ornaments which is shown in Figure 6. Other three dimensional ornaments have been created based on the obtained motifs shown in Figure 7 and Figure 8. Some motifs are
considered to be the pattern of animals and flowers which are mostly from hypocycloid and its variants due to some complex mappings. After becoming some ornaments, these ornaments are used to be decoration of Hypocycloid dance which is shown in youtube. The photos and some ornaments are shown in Figure 8 -9.

Visit on Hypocycloid

Deltoid, astroid and 5-cusp hypocycloid are particular hypocycloids created by positive integer values of *n=a/b* in Equation (1.2) for *n*= 3, 4 and *n*=5 respectively. Analysis the characteristics of the topological structures of existing planetary gear train type hypocycloid and epicycloid mechanisms with one degree of freedom has been shown (Bartolo, and Agustín,2008). Deltoid is mapped by
$f(z)=1/z+\bar{z}$ and $f(z)=z+iz^2$.

Figure 9. Above : Left to right : Motif obtained by *f*(*z*) = cos(*z*) where *z*-domain is the created ornaments are based on the motifs and used in Hypocycloid dance (on the dancers).

Figure 10. Left to right: Motif obtained by Deltoid and mapped by $f(z)=1/z+\bar{z}$ and $f(z)=z+iz^2$.

Finally, one may innovate the figure by increasing the number of *n* and plug the resulting curve of each *n* on the same figure. This is depicted in Figure 11. Note that doing this activity is becoming more easily done by programming than done by manual drawing as usually traditional mathematicians do. One may vary directly the obtained motifs by changing parameters though there exists no analytical explanation of the obtained motifs. Therefore knowing a bit of programming language in mathematics education will give higher performance of mathematics.

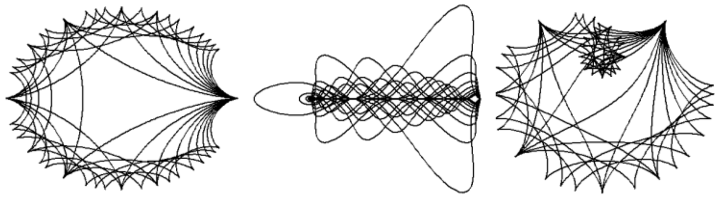

Figure 11. Left to right: Motifs obtained by Deltoid and mapped by $f(z)=1/z+\bar{z}$ and $f(z)=z+iz^2$ by varying *n*=2,...,10.

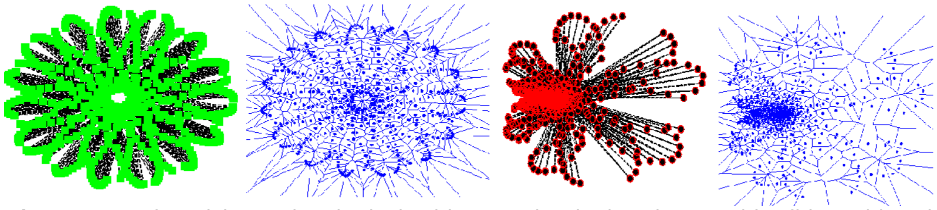

Figure 12. Left to right: Each pair obtained by Equation (1.2) and mapped by *f*(*z*)=cos(*z*) and both are mapped by voronoi mapping.

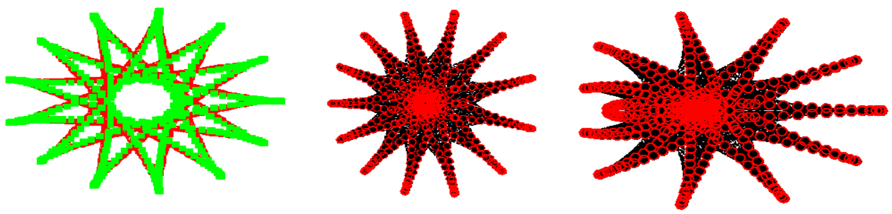

Figure 13. Left to right: hypocycloid with *a/b*=1.6, $0 \le t \le 200\pi$, its mapped by $f(z) = \cos z$ and its derivative of f(z), i.e $g(z) = -\sin z$.

As shown in theoretical part, one needs to visualize a derivative of a complex function. Let us consider *f*(*z*)=cos *z* and its derivative, i.e –sin z applied to the z-domain created by a hypocycloid curve with an irrational value of *a/b*. An example is depicted in Figure 13.

Result on Voronoi mapping

Design of voronoi mapping is not discussed into detail here since the research concerns only its output due to the given domain, i.e parametric curve and the result of complex mapping. Suppose, we have the hypocycloid equation shown Equation (1.2). By employing the f(z)=cos(z) to this equation, one has the real part and the imaginary part separately. We have two domains, i.e the parametric curve and the complex mapping for Voronoi mapping. The Voronoi function provided by MATLAB is applied to these two domains and one yields the Voronoi diagrams as shown in Figure 12. The obtained patterns are possible to be clothes motifs in future research.

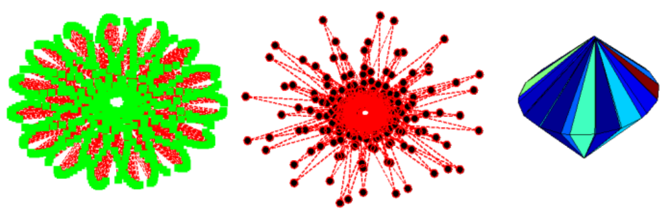

Figure 14. Left to right: Hypocycloid with $a = \pi, b = \sqrt{2}$, $a = \pi$, mapped by 1/z, 3-dimensional Delaunay tessellation of (*x,y,u*) where (*x,y*) taken from Hypocycloid and u is the real part of 1/z.

Creation on motifs and the resulting ornaments can be also improved in three dimensional objects, thanks to some user friendly functions that are available in MATLAB. Hypocycloid with irrational value of *a/b* mapped by 1/*z* will give sets of (*x,y,u*) and (*x,y,v*). These sets are possible to present using dual of Voronoi diagram given by delaunay3.m in MATLAB. One of examples is shown in Figure 14 (the third one from left to right). Some possible 3 dimensional objects can still be created.

CONCLUSION

Parametric curves such as hypocycloid, epicycloid and epitrochoid are reviewed here to be the z-domains. The resulting curves are then can be mapped by complex mapping and Voronoi mapping. In the case of complex mappings, one needs to express into real part an imaginary part for each function. Several complex functions are shown here, e.g the real part and imaginary parts of cos z, sin z,1/z, z^2, and e^z, $\frac{1+z}{1-z}$ The derivative of each function can also be derived and considered to be some mappings that are used to create other motifs. Since some motifs appear to be animals and flowers, these motifs are created to be pattern of ornaments for the Hypocycloid dance.

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